

# CONFORMAL IMMERSIONS OF PRESCRIBED MEAN CURVATURE IN $\mathbb{R}^3$

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**ABSTRACT.** We prove the existence of (branched) conformal immersions  $F : S^2 \rightarrow \mathbb{R}^3$  with mean curvature  $H > 0$  arbitrarily prescribed up to a 3-dimensional affine indeterminacy. A similar result is proved for the space forms  $\mathbb{S}^3$ ,  $\mathbb{H}^3$  and partial results for surfaces of higher genus.

## 1. INTRODUCTION

Consider an immersed closed surface

$$F : S \rightarrow \mathbb{R}^3,$$

in Euclidean space  $\mathbb{R}^3$ . In terms of metric or Riemannian geometry, the Cauchy data of the immersion  $F$  consist of the first and second fundamental forms  $(\gamma, A)$  of the surface  $\Sigma = \text{Im} F$ . A basic and natural question is what are the possible elliptic data that can be imposed on  $F$ , giving then a well-defined and geometric elliptic PDE problem for the immersion. Roughly speaking, elliptic data consist of one-half of the Cauchy data, so a combination of three of the six components of  $(\gamma, A)$ . (This of course matches the three scalar components of  $F$ ).

Clearly one should consider first the cases of prescribing the induced metric  $\gamma$  or the second fundamental form  $A$ . One can view  $\gamma$  as Dirichlet data and  $A$  as Neumann data for  $F$ . This becomes clearer when one considers  $\Sigma$  as the boundary of compact domain  $M \subset \mathbb{R}^3$  (or immersion of a domain  $M$  into  $\mathbb{R}^3$ ); the pair  $(\gamma, A)$  is then Dirichlet and Neumann data for a flat metric on  $M$  with  $\partial M = \Sigma$ .

Prescribing the induced metric  $\gamma$  is the well-known isometric immersion problem: given a metric  $\gamma$  on  $S$ , is there an isometric immersion  $F : S \rightarrow \mathbb{R}^3$ , i.e.  $F^*(g_{\text{Eucl}}) = \gamma$ ? This problem has remained notoriously difficult despite much effort. From a large-scale viewpoint, relatively little progress has been made on this global problem except in the case of positive Gauss curvature  $K > 0$ , i.e. the solution of the Weyl problem by Nirenberg [17] and Pogorelov [19].

However, prescribing the induced metric  $\gamma$  is not an elliptic problem. This follows easily from Gauss' Theorema Egregium,

$$(1.1) \quad K = \det A,$$

where  $K$  is the Gauss curvature. Namely, if prescribing a  $C^{m,\alpha}$  metric  $\gamma \in \text{Met}^{m,\alpha}(S)$  were elliptic, then by elliptic regularity,  $F$  would be a  $C^{m+1,\alpha}$  mapping and hence  $A$  would be a  $C^{m-1,\alpha}$  form on  $S$ . Then (1.1) gives  $K \in C^{m-1,\alpha}(S)$ . However, the space of  $C^{m,\alpha}$  metrics for which the curvature  $K$  is in  $C^{m-1,\alpha}$  is of infinite codimension, so that the operator  $F \rightarrow \gamma = F^*(g_{\text{Eucl}})$  is not Fredholm. This contradicts ellipticity. Similar reasoning shows that prescribing the second fundamental form  $A$  is never elliptic.

It is perhaps worth noting that isometric immersions can be described by an associated Darboux equation, cf. [9] for instance, which in case the relation  $K > 0$  holds is an elliptic equation of Monge-Ampere type. However, there is a loss of one derivative in passing to the Darboux equation, so there is no conflict with the lack of ellipticity of the immersion problem itself.

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Returning to the issue of elliptic data for the map  $F$ , it was shown in [2] that the data  $([\gamma], H)$ , where  $[\gamma]$  is the pointwise conformal class of the induced metric  $\gamma$  and  $H = H_F$  is the mean curvature of the immersion  $F$ , are elliptic data. In fact, to the author's knowledge, this is the only known elliptic data, depending only on the Cauchy data  $(\gamma, A)$ . The data  $([\gamma], H)$  form a nonlinear elliptic system of three equations in the three unknowns - the components of the mapping  $F$ . Note that  $[\gamma]$  involves the first derivatives of  $F$  while  $H$  involves second derivatives.

In this paper we study the question of global existence of solutions to this elliptic problem.

*Question.* Given a closed orientable surface  $S$  and arbitrary smooth data  $([\gamma], H)$  on  $S$ , does there exist an immersion  $F : S \rightarrow \mathbb{R}^3$  realizing  $([\gamma], H)$ , i.e.

$$([F^*(g_{Eucl}), H_F]) = ([\gamma], H).$$

In this generality, the answer is clearly no. For instance, for any immersion  $F$  of a compact surface in  $\mathbb{R}^3$ , one must have  $H_F > 0$  somewhere. There are in fact further obstructions, at least in the case  $S = S^2$ . Namely, the mean curvature  $H$  of a conformal immersion  $(S, [\gamma]) \rightarrow (\mathbb{R}^3, g_{Eucl})$  must satisfy

$$(1.2) \quad \int_S V(H) dV_\gamma = 0,$$

for any conformal vector field  $V$  on  $S$ , cf. Section 2 for a proof. Thus for instance if  $H$  is a monotone function of a standard height function  $z$  on  $S^2$ , then  $H$  cannot be realized as the mean curvature of any conformal immersion  $S^2 \rightarrow \mathbb{R}^3$ . Formally, (1.2) gives a 3-dimensional space of conditions on  $H$ , corresponding to the 3-dimensional space of (restrictions of) linear functions on  $S^2$ .

We mainly focus here on the case  $S = S^2$ ; a discussion of the case of higher genus is given in Section 4. To state the main result, let  $C_+^{m-1, \alpha}$  denote the space of  $C^{m-1, \alpha}$  functions  $H : S^2 \rightarrow \mathbb{R}^+$ , so  $H > 0$  everywhere, with  $m \geq 1, \alpha \in (0, 1)$ . Define an equivalence relation on  $C_+^{m-1, \alpha}$  by

$$(1.3) \quad [H_1] = [H_2] \Leftrightarrow H_2 = H_1 + \ell,$$

where  $\ell = a + bx$  and  $x$  is the restriction to  $S^2(1) \subset \mathbb{R}^3$  of a linear function on  $\mathbb{R}^3$  of norm 1 and  $a = |b| \geq 0$ . Thus  $\ell \geq 0$  on  $S^2$  with  $\ell = 0$  at the point  $-x \in S^2$ . In particular, the gradients  $\nabla \ell$  form the space of conformal vector fields on  $S^2$ . Let  $\mathcal{D}_+^{m-1, \alpha} = C_+^{m-1, \alpha} / \sim$  be the quotient space.

**Theorem 1.1.** *For any pointwise  $C^{m, \alpha}$  conformal class  $[\gamma]$  of metrics on  $S^2$  and for any equivalence class  $[H] \in \mathcal{D}_+^{m-1, \alpha}$ , there exists a  $C^{m+1, \alpha}$  branched immersion of  $S^2$  into  $\mathbb{R}^3$  with prescribed pointwise conformal class  $[\gamma]$  and prescribed mean curvature class  $[H]$ . Thus, there exists a  $C^{m+1, \alpha}$  smooth branched immersion  $F : (S^2, \gamma) \rightarrow (\mathbb{R}^3, g_{+1})$  such that*

$$(1.4) \quad [F^*(g_{Eucl})] = [\gamma] \quad \text{and} \quad H(F(x)) = H + \ell,$$

for any  $H > 0$  and for some affine function  $\ell$  as above on  $S^2$ .

We refer to Section 2 for the precise definition of branched immersion; (it is the usual definition). Branched immersions also satisfy the obstruction (1.2). Generically the data  $([\gamma], H)$  is realized by a (regular) immersion; the space of branched immersions corresponds to a “boundary” of the space of immersions which has codimension at least 6. On the other hand, there is no reason to expect that Theorem 1.1 is valid within the space of immersions itself. The presence of branch points corresponds to bubbling behavior, commonly arising in geometric PDE problems. Such bubble formation can be ruled out for “low-energy” solutions, corresponding here to the mean curvature  $H$  close to a constant, but is not to be expected to be ruled out in general; see also the discussion of examples in Section 2.

The literature of studies of surfaces of constant and more generally prescribed mean curvature in Euclidean space  $\mathbb{R}^3$  and the related space forms  $\mathbb{S}^3$  and  $\mathbb{R}^3$  is vast. Some of the issues discussed

here are related to a question posed by Yau [26]: given a smooth function  $H : \mathbb{R}^3 \rightarrow \mathbb{R}$ , is there an immersion  $F$  of a surface  $\Sigma \rightarrow \mathbb{R}^3$  such that the mean curvature of  $F(\Sigma)$  equals  $H$ , cf. [4], [7], [23] and further references therein for example. Theorem 1.1 deals with a closely related but nevertheless somewhat different problem. Also, this paper may be viewed as a continuation of an earlier study in [2].

Theorem 1.1 is proved in Section 3 by a global degree-theoretic argument, based on the degree of proper Fredholm maps between Banach manifolds of Fredholm index 0. In Section 2, we set up the basic framework and background, proving in particular that the space of branched immersions  $F : S^2 \rightarrow \mathbb{R}^3$  is a smooth Banach manifold, for which the map to the target data  $F \rightarrow ([\gamma], H)$  is smooth and Fredholm, of Fredholm index 0. In Section 3, we prove a basic apriori bound on the area of an immersion  $F$  in terms of the target data  $([\gamma], H)$ . This is the essential estimate used to obtain a proper Fredholm map. A computation of the degree then follows, based on the Hopf uniqueness theorem for constant mean curvature spheres immersed in  $\mathbb{R}^3$ .

Theorem 1.1 generalizes naturally to the other space forms  $\mathbb{S}^3$  and  $\mathbb{H}^3$ . This is discussed in Section 4, cf. Theorem 4.1. Also in Section 4, we point out that all of the results above carry over to closed orientable surfaces of any genus  $g > 0$ , except for the last result, namely the computation of the degree. This remains an interesting open question.

## 2. PRELIMINARY MATERIAL

In this section we describe background material and results needed to prove Theorem 1.1.

Let  $Map^{m+1,\alpha}(S^2, \mathbb{R}^3)$  denote the space of  $C^{m+1,\alpha}$  maps  $F : S^2 \rightarrow \mathbb{R}^3$ . Throughout the paper we assume  $m \geq 1$  and also allow  $m = \infty$ . This is a smooth Banach manifold, in fact Banach space due to the linear structure on  $\mathbb{R}^3$ ; when  $m = \infty$  one has a Fréchet space. Although  $Map^{m+1,\alpha}(S^2, \mathbb{R}^3)$  is not separable, it is separable with respect to a slightly weaker topology, namely the  $C^{m+1,\alpha'}$  topology, for any  $\alpha' < \alpha$ , cf. [24] for further discussion. The tangent space at a map  $F$  is given by the space of  $C^{m+1,\alpha}$  vector fields  $X$  along the map  $F$ .

Next, let  $Imm^{m+1,\alpha}(S^2, \mathbb{R}^3)$  be the space of  $C^{m+1,\alpha}$  immersions  $F : S^2 \rightarrow \mathbb{R}^3$ . This is an open domain in  $Map^{m+1,\alpha}(S^2, \mathbb{R}^3)$  and so of course is also a smooth Banach manifold. Let  $\Sigma = ImF$  denote the immersed sphere in  $\mathbb{R}^3$ .

Let  $\mathcal{C}^{m,\alpha} = C^{m,\alpha}(S^2)$  be the space of (pointwise) conformal classes  $[\gamma]$  of  $C^{m,\alpha}$  metrics on  $S^2$ ; recall that two metrics  $\gamma_1$  and  $\gamma_2$  on  $S^2$  are conformally equivalent if  $\gamma_2 = \mu^2 \gamma_1$ , for some positive function  $\mu$  on  $S^2$ . Let  $C^{m-1,\alpha} = C^{m-1,\alpha}(S^2)$  be the space of  $C^{m-1,\alpha}$  functions on  $S^2$ . An immersion  $F \in Imm^{m+1,\alpha}(S^2, \mathbb{R}^3)$  induces a metric  $\gamma = F^*(g_{Eucl})$  on  $S^2$  and mean curvature function  $H = H_F \in C^{m-1,\alpha}$ , with respect to a choice of normal. Here the normal is chosen to be that induced by the outward normal to the sphere tangent to  $\Sigma = ImF$  at a point where  $|F|^2$  is maximal.

This data may be assembled to a natural map of Banach manifolds

$$(2.1) \quad \Pi : Imm^{m+1,\alpha}(S^2, \mathbb{R}^3) \rightarrow \mathcal{C}^{m,\alpha} \times C^{m-1,\alpha},$$

$$\Pi(F) = ([\gamma], H).$$

By [1], [2], the data  $([\gamma], H)$  form an elliptic system of PDEs for the map  $F$ , cf. also the proof of Proposition 2.2 below. Thus the map  $\Pi$  is a Fredholm map, i.e. the linearization  $D\Pi$  at any  $F$  is a Fredholm linear map. However, the map  $\Pi$  has a rather large degeneracy, due to the large isometry group  $Isom(\mathbb{R}^3)$  of  $(\mathbb{R}^3, g_{Eucl})$ . This group acts freely on  $Imm \equiv Imm^{m+1,\alpha}(S^2, \mathbb{R}^3)$  via  $(F, \iota) \rightarrow \iota \circ F$ , corresponding to translation, rotation or reflection of  $F$  and fixes the target data, i.e.  $\Pi(\iota \circ F) = \Pi(F)$ . To remove this degeneracy, we divide  $Imm$  by this action, and consider only the quotient space  $Imm_b$  of based immersions. There is a global slice to this action, i.e. an inclusion  $Imm_b \subset Imm$ , given by fixing a point  $p_0 \in \mathbb{S}^2(1) \subset \mathbb{R}^3$  (say the north pole), a unit vector  $e \in T_{p_0}(\mathbb{S}^2(1))$  and requiring that  $F(p_0) = 0$ ,  $T_0(\Sigma) = \mathbb{R}^2 \subset \mathbb{R}^3$  and with  $F_*(e) = (a, 0, 0)$  for some

$a > 0$ . Thus, unless mentioned otherwise, throughout the paper we consider

$$(2.2) \quad \begin{aligned} \Pi : Imm_b^{m+1,\alpha}(S^2, \mathbb{R}^3) &\rightarrow \mathcal{C}^{m,\alpha} \times C^{m-1,\alpha}, \\ \Pi(F) &= ([\gamma], H). \end{aligned}$$

By [1], the Fredholm index of  $\Pi$  equals zero. (The Fredholm index of  $\Pi$  in (2.1) equals 6, the dimension of  $Isom(\mathbb{R}^3)$ ).

The basic issue in studying the global properties of the map  $\Pi$  in (2.2) is whether  $\Pi$  is proper. This is not true per se, due to the non-compactness of the conformal group of  $S^2$ . (This is another reason for dividing out by the action of isometries above). In fact, by the uniformization theorem, the group of  $C^{m+1,\alpha}$  diffeomorphisms of  $S^2$  acts transitively on the space  $\mathcal{C}^{m,\alpha}$ , with stabilizer the conformal group  $\text{Conf}(S^2)$  of  $(S^2, [\gamma])$ . This is a non-compact group, diffeomorphic to  $\mathbb{R}^3$ . The group  $\text{Conf}(S^2)$  also acts on the space of functions  $C^{m-1,\alpha}$  by pre-composition,  $(H, \varphi) \rightarrow H \circ \varphi$ , but here the action is proper, except at the constant functions  $H = c$ . It follows that for any point  $([\gamma], c) \in \mathcal{C}^{m,\alpha} \times C^{m-1,\alpha}$ , the space  $\Pi^{-1}(\mathcal{C}^{m,\alpha} \times \{c\})$  is non-compact, so that  $\Pi$  is not proper. On the other hand, the action of the conformal group on  $\mathcal{C}^{m,\alpha} \times C^{m-1,\alpha}$  is proper, whenever  $H$  is non-constant.

There are two ways to deal with this issue and we will in fact use both. First, one may simply remove the sets  $\Pi^{-1}([\gamma], c)$  above from the domain of  $\Pi$  in (2.2). Thus, let  $Imm_0^{m+1,\alpha}(S^2, \mathbb{R}^3) = \Pi^{-1}(\mathcal{C}^{m,\alpha} \times [C^{m-1,\alpha} \setminus \{\text{constants}\}])$ . This is the set of (based) immersions of  $S^2$  whose images are not spheres of constant mean curvature  $c$ . By the well-known Hopf theorem [10], these are just the round, constant curvature spheres  $S^2(r)$ . One may then consider

$$(2.3) \quad \begin{aligned} \Pi_0 : Imm_0^{m+1,\alpha}(S^2, \mathbb{R}^3) &\rightarrow \mathcal{C}^{m,\alpha} \times [C^{m-1,\alpha} \setminus \{\text{constants}\}], \\ \Pi_0(F) &= ([\gamma], H). \end{aligned}$$

For  $\Pi_0$ , conformal reparametrizations of a given immersion act properly on the target data.

Alternately, this problem may be remedied in the usual way by choosing a suitable 3-point marking for the mapping  $F$ . Thus, fix three points  $p_i$  on the round 2-sphere  $(\mathbb{S}^2(1), g_{+1}) \subset \mathbb{R}^3$  with

$$(2.4) \quad \text{dist}_{g_{+1}}(p_i, p_j) = \pi/2,$$

for  $i \neq j$ . Let  $Imm_1^{m+1,\alpha}$  be the submanifold of  $Imm_b^{m+1,\alpha}$  consisting of immersions  $F$  such that the pull-back metric  $\gamma = F^*(g_{Eucl})$  satisfies (2.4), (with  $\gamma$  in place of  $g_{+1}$ ). This is a submanifold of codimension 3, representing a slice to the action (by pre-composition) of the conformal group on  $Imm_b^{m+1,\alpha}$ . Restriction gives an induced map

$$(2.5) \quad \Pi' : Imm_1^{m+1,\alpha}(S^2, \mathbb{R}^3) \rightarrow \mathcal{C}^{m,\alpha} \times C^{m-1,\alpha}.$$

The map  $\Pi'$  is of course still Fredholm. However, while the map  $\Pi$  has Fredholm index 0,  $\Pi'$  now has Fredholm index -3. In order to obtain again a map of index 0, one needs to take a quotient of the target space by a 3-dimensional space transverse to the map  $\Pi'$ . To motivate this, we first examine the constraint equations, namely the Gauss-Codazzi and Gauss equations, relating the intrinsic and extrinsic geometry of the surface  $\Sigma = ImF$ . Thus, for an immersed surface  $\Sigma \subset \mathbb{R}^3$ , one has:

$$(2.6) \quad \delta(A - H\gamma) = -Ric_{g_{Eucl}}(N, \cdot) = 0,$$

$$(2.7) \quad |A|^2 - H^2 + R_\gamma = R_{g_{Eucl}} - 2Ric_{g_{Eucl}}(N, N) = 0.$$

Here  $A$  is the second fundamental form of the immersion  $F$  while  $R_\gamma$  and  $R_{g_{Eucl}}$  are the scalar curvatures of  $\gamma$  and  $g_{Eucl}$  respectively. The scalar constraint (2.7) will be important later, in Section 3.

An immediate consequence of the divergence constraint (2.6) is the following. Suppose  $V$  is a conformal Killing field on  $(S^2, \gamma)$ ,  $\gamma = F^*(g_{Eucl})$ . Then pulling back the geometric data on  $\Sigma$  back to  $S^2$  via the immersion  $F$ , one has

$$\int_{S^2} \langle V, \delta(A - H\gamma) \rangle dV_\gamma = \int_{S^2} \langle \delta^* V, A - H\gamma \rangle V_\gamma = -\frac{1}{2} \int_{S^2} H \operatorname{div} V dV_\gamma = \frac{1}{2} \int_{S^2} V(H) dV_\gamma.$$

Thus (2.6) gives the relation

$$(2.8) \quad \int_{S^2} V(H) dV_\gamma = 0,$$

for any conformal Killing field  $V$  on  $(S^2, \gamma)$ . In particular, as noted in the Introduction, the relation (2.8) gives an obstruction to the surjectivity of  $\Pi$  onto the space of mean curvature functions, somewhat analogous to the Kazdan-Warner type obstruction [12] for prescribed scalar curvature in a conformal class. For a given  $\gamma$ , (2.8) is a 3-dimensional restriction on the form of  $H$  for an immersion  $F$ . Note however that the constraint (2.8) is *not* a condition on the target space data  $([\gamma], H)$ , exactly due to presence of the volume form  $dV_\gamma$ .

As in the Introduction, on the target space  $C^{m-1, \alpha}$  of  $C^{m-1, \alpha}$  functions on  $S^2$ , define an equivalence relation by

$$(2.9) \quad [H_2] = [H_1] \Leftrightarrow H_2 = H_1 + \ell,$$

for some normalized affine function  $\ell$  on  $S^2(1) \subset \mathbb{R}^3$  as following (1.3). The space of such affine functions is exactly  $\mathbb{R}^3$ , with the origin corresponding to  $a = b = 0$  and unit sphere corresponding to the unit vector  $x \in S^2$ . This action of  $\mathbb{R}^3$  on  $C^{m-1, \alpha}$  is free. Let  $\mathcal{D}^{m-1, \alpha}$  be the space of equivalence classes and let

$$(2.10) \quad \pi : \mathcal{C}^{m, \alpha} \times C^{m-1, \alpha} \rightarrow \mathcal{C}^{m, \alpha} \times \mathcal{D}^{m-1, \alpha},$$

be the projection map, (equal to the identity on the first factor). The fibers of  $\pi$  are  $\mathbb{R}^3$ , (the space of normalized affine functions). Clearly, the quotient  $\mathcal{C}^{m, \alpha} \times \mathcal{D}^{m-1, \alpha}$  is also a smooth Banach manifold.

Observe that for any fixed volume form  $dV_\gamma$ , (2.8) determines a unique representative  $H \in [H]$ , for the equivalence relation (2.9); given any representative  $H' \in [H]$ , there is a unique normalized affine function  $\ell$  such that  $H' + \ell$  satisfies (2.8).

By composing  $\Pi'$  in (2.5) with  $\pi$  above, we obtain

$$(2.11) \quad \Pi_1 : \operatorname{Imm}_1^{m+1, \alpha}(S^2, \mathbb{R}^3) \rightarrow \mathcal{C}^{m, \alpha} \times \mathcal{D}^{m-1, \alpha},$$

$$\Pi_1(F) = ([\gamma], [H]).$$

This is a smooth Fredholm map of Banach manifolds. We claim that the Fredholm index of  $\Pi_1$  is zero,

$$(2.12) \quad \operatorname{index}(D\Pi_1) = 0.$$

To prove this, it suffices to consider the linearization  $D\Pi_1$  at the standard embedding  $F_0 : S^2(1) \subset \mathbb{R}^3$ , since the index is invariant under deformations. The Fredholm index is 0 if the map  $\Pi'$  in (2.5) is transverse to the fibers of  $\pi$  in (2.10) at the standard embedding  $F_0$ . Thus suppose  $X$  is a deformation of the standard embedding  $F_0$ . The induced variation of the target data is given by  $([\delta^* X], [H'_X])$ . To prove transversality, we must show that if  $([\delta^* X], H'_X) = (0, \ell)$ , i.e. the variation is tangent to the fiber, then  $\ell = 0$ , i.e. the variation vanishes in the full target space.

To see this, decompose  $X = X^T + fN$  into tangential and normal components to  $\Sigma = \operatorname{Im} F$ . One then has, at  $F_0$ ,  $2H'_{\delta^* X} = -\Delta f - |A|^2 f + X^T(H) = -\Delta f - 2f$ , since  $H = |A|^2 = 2$ . Suppose

then  $L(f) = -\Delta f - 2f = \ell$ , for some  $\ell$ . Now  $\ell - a$  is a first eigenfunction of the Laplacian on  $\mathbb{S}^2(1)$ , so that  $L(\ell) = L(a) = -2a$ . Hence

$$\int_{S^2} \ell^2 dV = \int_{S^2} \langle L(f), \ell \rangle = \int_{S^2} \langle f, L(\ell) \rangle = -2a \int f.$$

But  $\int f = -\frac{1}{2} \int \ell = -2\pi a$ , which gives

$$\int_{S^2} \ell^2 = 4\pi a^2.$$

However, since  $\ell = a + bx$ ,  $\int \ell^2 = 4\pi a^2 + b^2 \int x^2$ . It follows that  $b = 0$  and hence  $a = 0$ , so that  $\ell = 0$ , proving the claim.

We will work with both maps  $\Pi_0$  in (2.3) and  $\Pi_1$  in (2.11) in the following. However, these maps are still not proper, now however for somewhat more subtle reasons than before. The basic reason is that immersions may converge to branched immersions maintaining control on the data  $([\gamma], H)$ . It is worthwhile to first illustrate this clearly on a concrete example.

Thus, consider Enneper's surface  $E$  in  $\mathbb{R}^3$ . This is a complete minimal surface immersed in  $\mathbb{R}^3$ , conformally equivalent to the flat plane  $\mathbb{R}^2$ . The immersion  $E : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  is given explicitly by:

$$\begin{aligned} x_1 &= u(v^2 - \frac{1}{3}u^2 + 1), \\ x_2 &= -v(u^2 - \frac{1}{3}v^2 + 1), \\ x_3 &= u^2 - v^2. \end{aligned}$$

Equivalently, for  $\zeta = u + iv$ ,  $x_1 = \operatorname{Re}(\zeta - \frac{1}{3}\zeta^3)$ ,  $x_2 = \operatorname{Im}(\zeta + \frac{1}{3}\zeta^3)$ ,  $x_3 = \operatorname{Re}(\zeta^2)$ . We now blow-down  $E$  and consider the limit, (the tangent cone at infinity). Thus, choose  $T$  large and replace  $(u, v)$  by  $(Tu, Tv)$ . Dividing both sides of the equations above by  $T^3$  and relabeling gives

$$\begin{aligned} x_1 &= u(v^2 - \frac{1}{3}u^2 + T^{-2}), \\ x_2 &= -v(u^2 - \frac{1}{3}v^2 + T^{-2}), \\ x_3 &= T^{-1}(u^2 - v^2). \end{aligned}$$

Setting  $t = T^{-1}$ , the curve of blow-downs  $E_t$  of the Enneper surface is given by

$$\begin{aligned} x_1 &= u(v^2 - \frac{1}{3}u^2 + t^2), \\ x_2 &= -v(u^2 - \frac{1}{3}v^2 + t^2), \\ x_3 &= t(u^2 - v^2). \end{aligned}$$

The immersions  $E_t$ ,  $t > 0$ , are conformal immersions  $\mathbb{R}^2 \rightarrow \mathbb{R}^3$  with  $H_{E_t} = 0$ . When  $t = 0$ ,  $E_0$  is the plane  $\mathbb{R}^2 = \{x_3 = 0\}$ , with multiplicity 3, so a 3-fold branched cover of  $\mathbb{R}^2$ . Taking the  $t$ -derivative gives the vector field

$$X = (2tu, 2tv, u^2 - v^2).$$

The curve of mappings  $E_t$  is smooth in  $t$ , for  $t \in \mathbb{R}$ . Note that when  $t < 0$  one obtains a reflection of the "original" Enneper surfaces  $E_t$  through the plane  $x_3 = 0$ .

While the behavior above takes place on the non-compact domain  $\mathbb{R}^2$ , it can be localized to a finite region, then suitably "bent" and extended to a curve of immersions  $F_t : S^2 \rightarrow \mathbb{R}^3$  with  $H_{F_t} > 0$  everywhere. Note in particular that the immersions  $E_t$  (or suitably modified  $F_t$ ) are uniformly bounded, in fact converge, in the  $C^\infty$  norm and similarly the mean curvatures  $H_{E_t}$  or  $H_{F_t}$  converge in  $C^\infty$  as  $t \rightarrow 0$ . In other words, the target data  $(\gamma_t, H_t)$  remain bounded (and

converge) while the family  $E_t$  or  $F_t$  does not limit on an immersion. This shows that  $\Pi$  is not proper on the space of immersions. Of course the Gauss curvature  $K$ , and the norm of the second fundamental form  $A$ , blow up as  $t \rightarrow 0$ .

Similar behavior occurs with the immersions  $x_1 = \operatorname{Re}(\zeta - \frac{1}{2k+1}\zeta^{2k+1})$ ,  $x_2 = \operatorname{Im}(\zeta + \frac{1}{2k+1}\zeta^{2k+1})$ ,  $x_3 = \operatorname{Re}(\frac{2}{k+1}\zeta^{k+1})$ , coming from the Weierstrass representation of (certain) minimally immersed planes in  $\mathbb{R}^3$ . Here one has a branch point of order  $2k+1$  at the origin in the blow-down limit. To obtain branch points of even order, in the Weierstrass representation, take  $f = z$ ,  $g = z^k$ . Then  $x_1 = \operatorname{Re}(\frac{1}{2}\zeta^2 - \frac{1}{2k+2}\zeta^{2k+2})$ ,  $x_2 = \operatorname{Im}(\frac{1}{2}\zeta^2 + \frac{1}{2k+2}\zeta^{2k+2})$ ,  $x_3 = \operatorname{Re}(\frac{2}{k+2}\zeta^{k+2})$ . This gives a branch point of even order  $2(k+1)$  in the blow-down limit. Note however that this surface is itself a branched minimal immersion; setting  $w = \zeta^2$  shows that it is a 2-fold branched cover of an immersion.

In sum, the construction above can be carried out on any complete minimally immersed plane in  $\mathbb{R}^3$  of finite total curvature.

We now begin a more precise discussion of the space of branched immersions, cf. [3], [8], [6] for further background. Let  $S^2$  be given its standard two charts, via stereographic projection, with local complex coordinate  $z = u + iv$  in each chart. This data is fixed throughout the following.

Recall that by the uniformization theorem, any immersion  $F \in \operatorname{Imm}^{m+1,\alpha}(S^2, \mathbb{R}^3)$  can be reparametrized to a conformal immersion, i.e. there is a diffeomorphism  $\varphi : S^2 \rightarrow S^2$  such that  $F \circ \varphi$  is conformal.

*Definition.* A map  $F \in C^{m+1,\alpha}(S^2, \mathbb{R}^3)$  is a conformal branched immersion if it is a conformal immersion away from finitely many singular points  $\{q_j\}$  and in a neighborhood  $D$  of each singular point  $q \in \{q_j\}$ , the complex gradient  $F_z = \frac{dF}{dz} = \frac{1}{2}(F_u - iF_v) : D \rightarrow \mathbb{C}^3$  satisfies

$$(2.13) \quad F_z = z^k G,$$

in the normalization  $z(q) = 0$ , where  $G : D \rightarrow \mathbb{C}^3$  is  $C^{m,\alpha}$  smooth and satisfies

$$(2.14) \quad G \cdot G = 0, \quad G(q) \neq 0.$$

Also  $k \geq 1$ ; the case  $k = 0$  corresponds to a neighborhood where  $F$  is an immersion.

A map  $F \in C^{m+1,\alpha}(S^2, \mathbb{R}^3)$  is a branched immersion if it is a reparametrization of a conformal branched immersion, so that  $F = \tilde{F} \circ \varphi$ , where  $\tilde{F}$  is a conformal branched immersion and  $\varphi$  is a diffeomorphism  $S^2 \rightarrow S^2$ .

The first condition in (2.14) is equivalent to the relation  $F_z \cdot F_z = 0$ , i.e.

$$F_u \cdot F_u - F_v \cdot F_v - 2iF_u \cdot F_v = 0,$$

so that  $F$  is conformal. The second condition in (2.14) then implies that the real and imaginary parts of  $G$  are linearly independent, and so span the tangent space  $T_{F(q)}\Sigma$  to the surface  $\Sigma = \operatorname{Im} F$ .

The mapping  $F$  may be recovered from (2.13) by solving the inhomogeneous Cauchy-Riemann equation, i.e.

$$F(z, \bar{z}) = \frac{1}{2\pi i} \int_D \bar{\xi}^k \bar{G}(\xi, \bar{\xi}) \frac{dz \wedge d\bar{z}}{\xi - z} + a(\bar{z}),$$

where  $a$  is anti-holomorphic. Since  $F$  is real-valued,  $a(\bar{z})$  is uniquely determined, (up to a constant). Observe that elliptic regularity implies that if  $G \in C^{m,\alpha}$  then  $F \in C^{m+1,\alpha}$ .

Let  $B\operatorname{Imm}^{m+1,\alpha}(S^2, \mathbb{R}^3)$  be the set of all branched immersions. This is viewed as a subspace of the Banach manifold  $\operatorname{Map}^{m+1,\alpha}(S^2, \mathbb{R}^3)$  and so is given the induced topology.

**Lemma 2.1.** *The space  $B\operatorname{Imm}^{m+1,\alpha}(S^2, \mathbb{R}^3)$  is a smooth Banach manifold.*

**Proof:** This is clear for a neighborhood of an immersion  $F \in Imm^{m+1,\alpha}(S^2, \mathbb{R}^3)$ . Suppose first  $F$  is a conformal branched immersion with singular points  $q_j$  of the form (2.13). Near any such singular point  $q$ , any conformal branched immersion  $\tilde{F}$  near  $F$  has the form

$$(2.15) \quad \tilde{F}_z = P^j(z)\tilde{G}^j + \nu N, \quad j = 1, 2, 3,$$

where  $P^j(z)$  is a polynomial of degree  $k$  near  $z^k$ ,  $N$  is the normal vector field to the immersion  $F$ ,  $\tilde{G}$  is near  $G$  and  $\nu$  is  $C^{m,\alpha}$  smooth with  $\nu(q) \sim 0$ . Of course one requires

$$\tilde{F}_z \cdot \tilde{F}_z = 0.$$

For example, if the polynomials  $P^j = P$  are all equal, with  $P = \prod_{i=1}^{\ell} (z - z_i)^{k_i}$ , then  $\tilde{F}$  is a nearby conformal branched immersion, with branch points of order  $k_i$  at  $z_i \sim 0$ . Of course  $\sum_1^{\ell} k_i = k$ . If on the other hand the three polynomials  $P^j$  have no zeros in common, then  $\tilde{F}$  is a regular conformal immersion near  $F$ . (This is the case for instance for the curve of Enneper immersions  $E_t$  above with  $t \neq 0$ ). If  $F$  has a branch point of order  $k$ , any *sufficiently* near branched immersion has branch points of order  $\leq k$ ; the order cannot jump up in sufficiently small neighborhoods. One sees here how curves of immersions pass smoothly to branched immersions by allowing distinct roots of the polynomials  $P^j$  to merge together.

The tangent space to the space of conformal branched immersions at a conformal branched immersion  $F$  consists of  $C^{m+1,\alpha}$  vector fields  $X$  along  $F$ . Away from branch points, the field  $X = dF_t/dt$  is  $C^{m+1,\alpha}$  and satisfies the linearization of (2.14), i.e.

$$(2.16) \quad X_z \cdot F_z = 0,$$

but is otherwise arbitrary. Near a branch point  $q$ ,  $X$  has the form

$$(2.17) \quad X_z = (P^j)'G^j + z^k G' + \nu' N,$$

with  $G' \in Map^{m,\alpha}(S^2, \mathbb{C}^3)$ . Since  $P_t^j = \prod_{i=1}^{\ell} (z - z_i^j(t))^{k_i}$ , one has  $(P^j)' = -z^{k-1} \sum_i k_i (z_i^j)'$ . The linearized equation (2.16) thus becomes

$$z^k [-z^{k-1} \sum_i k_i (z_i^j)' G^j \cdot G + z^k G' \cdot G] = 0,$$

near  $q$ , since  $G \cdot N = 0$ . Thus,  $X$  is tangent to the space of branched conformal immersions if and only if

$$(2.18) \quad \sum_i k_i (z_i^j)' = 0, \quad \text{and} \quad G' \cdot G = 0,$$

so that

$$(2.19) \quad X_z = z^k G' + \nu' N.$$

The first condition in (2.18) is a “balancing condition” on the tangent vectors to the curve of roots of  $P_t^j$  at  $z = 0$ . There is no condition on  $\nu'$  besides smoothness. Conversely, any  $X$  satisfying (2.18) is tangent to a curve  $F_t$  of conformal branched immersions with  $F_0 = F$ . This follows from the inverse function theorem, since the map  $G' \rightarrow G' \cdot G$  is surjective onto  $C^{m,\alpha}(D, \mathbb{C})$ .

Finally, the space  $BImm^{m+1,\alpha}(S^2, \mathbb{R}^3)$  consists of arbitrary  $C^{m+1,\alpha}$  smooth reparametrizations of conformal branched immersions, i.e. maps of the form  $F \circ \varphi$ , where  $F$  is a conformal branched immersion and  $\varphi$  is a  $C^{m+1,\alpha}$  diffeomorphism of  $S^2$ . Since both factors  $F$  and  $\varphi$  have Banach manifold structures, so does the full space of compositions. This completes the proof. ■

Note that the space  $BImm^{m+1,\alpha}(S^2, \mathbb{R}^3)$  is stratified by submanifolds according to branching orders. The strata corresponding to branched immersions with total branching order  $k$  has codimension  $6k = \dim \mathbb{C}^{3k}$ .



The mean curvature  $H$  of a branched immersion is not well-defined in general at the branch points. In fact, for a conformal branched immersion  $F$ , one has the formula,  $\frac{1}{4}\Delta_0 F = H(F_u \times F_v)$  where  $\Delta_0 = F_{uu} + F_{vv}$  is the flat Laplacian, (cf. [6] for instance). In terms of the complex gradient, this is equivalent to

$$(2.20) \quad F_{\bar{z}z} = iH(\bar{F}_z \times F_z).$$

One has

$$\bar{F}_z \times F_z = |z|^{2k} \bar{G} \times G,$$

and so for the mean curvature to be well-defined in  $C^{m-1,\alpha}$  one needs

$$(2.21) \quad \frac{F_{\bar{z}z}}{|z|^{2k}} \in C^{m-1,\alpha},$$

in a neighborhood of a branch point of order  $k$ . (Of course this holds automatically away from branch points). Conversely, by elliptic regularity associated to (2.20), if  $H \in C^{m-1,\alpha}$ , then  $F \in C^{m+1,\alpha}$ . Since  $F_z = z^k G$ , so  $F_{\bar{z}z} = z^k G_{\bar{z}}$  it is more convenient to express the equation above as

$$G_{\bar{z}} = \bar{z}^k \varphi N,$$

for some  $\varphi \in C^{m-1,\alpha}(S^2, \mathbb{C}^3)$ , where  $N$  is a unit normal for the surface  $\Sigma = \text{Im}F$ . Note that  $\bar{F}_z \times F_z$  and  $\bar{G} \times G$  are multiples of  $N$ .

*Definition.* A  $C^{m+1,\alpha}$  conformal branched immersion  $F$  is  $H$ -regular if near any branch point  $q$  of order  $k$  one has

$$(2.22) \quad G_{\bar{z}} = \bar{z}^k \varphi N,$$

for some  $\varphi \in C^{m-1,\alpha}(D, \mathbb{C})$ . A  $C^{m+1,\alpha}$  branched immersion  $F$  is  $H$ -regular if it is a reparametrization (by a  $C^{m+1,\alpha}$  diffeomorphism) of a conformal branched  $H$ -regular immersion.

The mean curvature  $H_F$  of an  $H$ -regular branched immersion is well-defined, and in  $C^{m-1,\alpha}(S^2)$ . The space of  $H$ -regular branched immersions is denoted by  $\mathcal{H}Imm^{m+1,\alpha}(S^2, \mathbb{R}^3)$  and is a closed submanifold of  $BImm^{m+1,\alpha}(S^2, \mathbb{R}^3)$ . In particular  $\mathcal{H}Imm^{m+1,\alpha}(S^2, \mathbb{R}^3)$  is itself a smooth Banach manifold.

The linearization of the mean curvature  $H$  among conformal branched immersions is given by

$$(2.23) \quad G'_{\bar{z}} = \bar{z}^k (iH'_X \bar{G} \times G + iH(\bar{G}' \times G + \bar{G} \times G')),$$

so that the linearization of (2.22) holds with  $\varphi' \in C^{m-1,\alpha}$ .

Next we also need to enlarge the target manifold  $C^{m,\alpha}$ . First, the metric  $\gamma$  on  $S^2$  is written in terms of the coordinates  $(z, \bar{z})$  in place of the real and imaginary parts of  $z$ . In the following, it is convenient to let  $x = u + iv = u_1 + iu_2$ . Thus

$$(2.24) \quad \gamma = \gamma_{zz} dz^2 + 2\gamma_{\bar{z}z} d\bar{z}dz + \gamma_{\bar{z}\bar{z}} d\bar{z}^2.$$

In terms of real coordinates  $\gamma = \gamma_{ij} du_i du_j$ , one has the relations

$$\gamma_{zz} = \frac{1}{4}(\gamma_{11} - \gamma_{22} - 2i\gamma_{12}), \quad \gamma_{\bar{z}z} = \frac{1}{2}(\gamma_{11} + \gamma_{22}), \quad \gamma_{\bar{z}\bar{z}} = \bar{\gamma}_{zz}.$$

Now let  $Met_s^{m,\alpha}(S^2)$  be the space of  $C^{m,\alpha}$  symmetric bilinear forms  $\gamma$  on  $S^2$  which are positive definite outside a finite number of singular points  $q_j$  and which near each singular point  $q \in \{q_j\}$ , have the form (2.24) with

$$\gamma_{zz} = z^{2k} \varphi_1, \quad \gamma_{\bar{z}z} = |z|^{2k} \varphi_2,$$

where  $\varphi_1 \in C^{m,\alpha}(D, \mathbb{C})$  and  $\varphi_2 \in C^{m,\alpha}(D, \mathbb{R}^+)$  near  $q$ .

Clearly  $Met_s^{m,\alpha}(S^2)$  is a smooth Banach manifold. The tangent space consists of  $C^{m,\alpha}$  symmetric bilinear forms  $h$  of the form (2.24) (no longer necessarily positive definite) with

$$(2.25) \quad h_{zz} = z^{2k} \varphi'_1, \quad h_{\bar{z}z} = |z|^{2k} \varphi'_2.$$

Define an equivalence relation on  $Met_s^{m,\alpha}(S^2)$  by setting  $\gamma_2 \sim \gamma_1$  if  $\gamma_2 = \mu^2 \gamma_1$ , for some positive function  $\mu \in C^{m,\alpha}(S^2)$ . Abusing notation, let  $\mathcal{C}^{m,\alpha}$  denote the quotient space. This is the space of pointwise conformal equivalence classes of singular, branched metrics on  $S^2$ .

Now given an  $H$ -regular branched immersion  $F \in \mathcal{HImm}^{m+1,\alpha}(S^2, \mathbb{R}^3)$ , the induced or pullback metric  $F^*(g_{Eucl})$  is a singular branched metric on  $S^2$ , so gives an element in  $Met_s^{m,\alpha}(S^2)$ . Similarly, the mean curvature  $H_F$  is a  $C^{m-1,\alpha}$  smooth function on  $S^2$ . Thus, one has a map

$$(2.26) \quad \begin{aligned} \Pi_0 : \mathcal{HImm}^{m+1,\alpha}(S^2, \mathbb{R}^3) &\rightarrow \mathcal{C}^{m,\alpha} \times [C^{m-1,\alpha}(S^2) \setminus \{\text{constants}\}], \\ \Pi_0(F) &= ([F^*(g_{Eucl})], H_F), \end{aligned}$$

extending the map  $\Pi_0$  in (2.3). From its construction,  $\Pi_0$  is a smooth map of Banach manifolds. Similarly, the map

$$(2.27) \quad \begin{aligned} \Pi_1 : \mathcal{HImm}^{m+1,\alpha}(S^2, \mathbb{R}^3) &\rightarrow \mathcal{C}^{m,\alpha} \times \mathcal{D}^{m-1,\alpha}, \\ \Pi_1(F) &= ([F^*(g_{Eucl})], [H_F]), \end{aligned}$$

extending the map  $\Pi_1$  in (2.11) is a smooth map of Banach manifolds.

**Proposition 2.2.** *The map  $\Pi_0$  in (2.26) is Fredholm, of Fredholm index 0.*

**Proof:** This is proved in [1], [2] when  $\Pi_0$  is restricted to regular immersions, as in (2.3). It is useful present the short proof here.

The linearization  $D\Pi_0$  acts on vector fields  $X$  along the immersion  $F$ . Write  $X = X^T + fN$ , where  $X^T$  is tangent and  $N$  is normal to  $\Sigma = Im(F)$ . Then

$$(2.28) \quad \delta^* X = \delta^*(X^T) + fA + df \cdot N,$$

so that  $(\delta^* X)^T = \delta^*(X^T) + fA$ . The second term here is lower order in  $X$  and so does not contribute to the principal symbol. The principal symbol  $\sigma$  of the first component of  $D\Pi_0$  is thus

$$(2.29) \quad \sigma([2(\delta^* X)^T]_0) = \sigma([2(\delta^*(X^T))]_0) = 2(\xi_i X_j - \frac{\xi_i X_i}{2} \delta_{ij}),$$

where  $i, j$  are indices along  $\Sigma$ , i.e. tangent to  $S^2$ . Setting this to 0 gives

$$\xi_1 X_2 = \xi_2 X_1 = 0 \quad \text{and} \quad \xi_1 X_1 = \xi_2 X_2.$$

Since  $(\xi_1, \xi_2) \neq (0, 0)$ , it is elementary to see that the only solution of these equations is  $X_1 = X_2 = 0$ . Next, for the mean curvature, one has  $2H'_{\delta^* X} = -\Delta f - |A|^2 f + X^T(H)$ . Here  $\Delta$  is the Laplacian with respect to the induced metric  $\gamma = F^*(g_{Eucl})$ . Hence  $\Delta = \lambda^{-2} \Delta_0$ , for  $\gamma = \lambda^2 g_0$ . The leading order symbol acting on  $f$  is thus  $\lambda^{-2} |\xi|^2 f$ . For a regular immersion,  $\lambda > 0$  and the vanishing of this term implies  $f = 0$ . Thus, the symbol of  $D\Pi_0$  is elliptic, so that by the regularity theory for elliptic systems, cf. [16] for instance,  $D\Pi_0$  is Fredholm. Note that the operator  $D\Pi_0$  has terms containing both first and second derivatives of  $X$ .

The proof at or near branch points of  $F$  is very similar, and we carry out the details below. Without loss of generality, we may assume that  $F$  itself is conformal.

For  $F_z = \frac{1}{2}(F_u - iF_v)$  and  $F_{\bar{z}} = \bar{F}_z = \frac{1}{2}(F_u + iF_v)$  one has

$$4F_z \cdot F_z = F_u \cdot F_u - F_v \cdot F_v - 2iF_u \cdot F_v,$$

so that  $F$  conformal if and only if

$$F_z \cdot F_z = 0.$$

Similarly

$$4F_z \cdot F_{\bar{z}} = F_u \cdot F_u + F_v \cdot F_v \equiv \lambda^2,$$

with  $\lambda^2 \in C^{m,\alpha}(S^2, \mathbb{R})$ . Thus, the non-conformal variation of the metric is given by

$$\frac{d}{dt} F_z^t \cdot F_z^t = 2X_z \cdot F_z,$$

where  $X = dF_t/dt$ . From (2.19) one then has

$$(2.30) \quad X_z \cdot F_z = z^{2k} G' \cdot G.$$

This is exactly of the form (2.25) with

$$\varphi'_1 = G' \cdot G.$$

One may choose a basis of  $\mathbb{R}^3$  so that  $G^3 = 0$  so that the tangent space to  $\Sigma = \text{Im}F$  at  $F(q) = 0$  is the  $(x_1, x_2)$  plane  $\mathbb{R}^2 \subset \mathbb{R}^3$ . Then (2.30) is clearly Fredholm in  $(G^j)'$  for  $j = 1, 2$  and so Fredholm in the first two components of  $X_z$ , (just as above following (2.29)).

Next, as in (2.23), the linearization  $H'$  of the mean curvature is given by

$$G'_z = \bar{z}^k (iH'_X \bar{G} \times G + iH(\bar{G}' \times G + \bar{G} \times G')).$$

As before, given  $(G^j)'$ ,  $j = 1, 2$ , this gives an elliptic equation for the remaining third component  $(G^3)'$  of  $G'$ , or equivalently an elliptic equation for the third component of  $X_z$ . It follows that  $D\Pi_0$  is Fredholm. By the deformation invariance of the index,  $D\Pi_0$  is of Fredholm index 0. ■

**Proposition 2.3.** *The map  $\Pi_1$  in (2.27) is Fredholm, of Fredholm index 0.*

**Proof:** This follows directly from Proposition 2.2 and (2.12). ■

### 3. PROPERNESS

In this section, we prove that the maps  $\Pi_0$  and  $\Pi_1$  in (2.26) and (2.27) are proper, when restricted to the domains where  $H_F > 0$ . Given Propositions 2.2 and 2.3, this is the key to the proof of Theorem 1.1.

Proving that  $\Pi_0$  is proper requires showing that an immersion (or branched immersion)  $F : S^2 \rightarrow \mathbb{R}^3$  is controlled in  $C^{m+1, \alpha}$  by the target data  $([F^*(g_{Eucl}), H_F])$ . The starting point to obtain such control is the Gauss constraint (2.7). Thus, integrating (2.7) along the immersion  $F$  and using the Gauss-Bonnet theorem gives

$$(3.1) \quad \int_{S^2} |A|^2 = \int_{S^2} H^2 - 4\pi\chi(S^2) = \int_{S^2} H^2 - 8\pi.$$

Since  $H$  is pointwise controlled by the target data, (3.1) gives control on the  $L^2$  norm of  $A$ , provided one can obtain a bound on the area of  $\Sigma = \text{Im}F$ . Note this does not follow directly from control of the conformal class  $[\gamma]$ .

Let  $F_t$  be a curve in  $\mathcal{HImm}^{m+1, \alpha}(S^2, \mathbb{R}^3)$  with  $F_0$  a fixed embedding - say near the standard round sphere. Although the mean curvature  $H$  controls the variation of the area of  $\Sigma_t = \text{Im}F_t$  in the normal direction, pointwise control of  $H$  alone does not give rise to an area bound. For example, consider a long cylinder  $C = [-L, L] \times S^1(1)$  embedded in  $\mathbb{R}^3$ , of mean curvature  $H = 1$ . One may attach two spherical caps to the boundary  $\partial C$  to obtain an embedding  $F_L : S^2 \subset \mathbb{R}^3$  with  $1 \leq H \leq 2$  pointwise. As  $L \rightarrow \infty$ ,

$$\text{area}(\Sigma_L) \rightarrow \infty,$$

with uniform control on  $H$ ; here  $\Sigma_L = \text{Im}F_L$ . Performing a similar construction with the periodic Delaunay cylinders of constant mean curvature, one may find curves of immersions  $F_t$  with the same behavior with

$$1 \leq H \leq 1 + \varepsilon,$$

for any fixed  $\varepsilon > 0$ . It is important to note however that the (pointwise) conformal structure of  $(S^2, \gamma_t)$ ,  $\gamma_t = F_t^*(g_{Eucl})$  (or  $\gamma_L = F_L^*(g_{Eucl})$ ) degenerates, i.e. diverges to infinity, in the examples

above. In particular, the mean curvature viewed as a function  $H : S^2 \rightarrow \mathbb{R}$  with a fixed (standard) atlas on  $S^2$  does not remain uniformly bounded in  $C^{m-1,\alpha}$  in these examples.

We first prove the area estimate for  $\Pi_0$ , and then show a similar argument gives the result for  $\Pi_1$ . Let  $\mathcal{H}Imm_+^{m+1,\alpha} = \mathcal{H}Imm_+^{m+1,\alpha}(S^2, \mathbb{R}^3)$  be the space of  $H$ -regular branched immersions with mean curvature  $H_F > 0$ .

**Proposition 3.1.** *For  $F \in \mathcal{H}Imm_+^{m+1,\alpha}(S^2, \mathbb{R}^3)$ , suppose the target data  $([\gamma], H) = ([F^*(g_{Eucl})], H_F)$  satisfy*

$$(3.2) \quad \|([\gamma], H)\| \leq L,$$

where the norm is taken in the normed target space  $\mathcal{C}^{m,\alpha} \times C_+^{m-1,\alpha}$ . Suppose also

$$(3.3) \quad \|H - c\|_{C^{m-1,\alpha}} \geq L^{-1},$$

for any constant  $c$ .

Then there is a constant  $A_0$ , depending only on  $L$ , such that

$$(3.4) \quad \text{area}(F) \leq A_0.$$

**Proof:** Since any branched immersion can be perturbed slightly to an immersion with a correspondingly small effect on the target data  $([\gamma], H)$ , it suffices to prove the result for  $F \in Imm_+^{m+1,\alpha}(S^2, \mathbb{R}^3)$ . By a well-known result of Smale [21], the space  $Imm_+^{m+1,\alpha}(S^2, \mathbb{R}^3)$  is connected. Moreover, it then follows from [15] that  $Imm_+^{m+1,\alpha}(S^2, \mathbb{R}^3)$  is connected.

Fix an embedding  $F_0 : S^2 \rightarrow \mathbb{R}^3$  near that of the standard round sphere. Let  $F_t$ ,  $0 \leq t \leq L$  be a curve in  $Imm_+^{m+1,\alpha}$  starting at  $F_0$  and ending at  $F_L = F$ . We may assume that  $F_t$  is parametrized in such a way that the image curve  $\Pi(F_t)$  is parametrized by (or proportional to) arclength, so that (3.2)-(3.3) hold along  $F_t$ .

Let  $X = dF_t/dt$  be the variation vector field of the curve  $F_t$ , and let  $\kappa = (\mathcal{L}_X g)^T = 2(\delta^* X)^T$  be the variation of the induced metric  $\gamma$ . Write  $X = X^T + fN$ , so that  $\frac{1}{2}\kappa = \delta^* X^T + fA$ . By the standard second variation formula for area, the variation of the mean curvature is given by

$$(3.5) \quad H'_X = -\Delta f - |A|^2 f + X^T(H).$$

Integrating this over  $S^2$  gives

$$(3.6) \quad \int_{S^2} H'_X = \int_{S^2} -|A|^2 f + X^T(H).$$

By the uniformization theorem, any metric  $\gamma$  on  $S^2$  is of the form  $\gamma = \lambda^2 \psi^*(\gamma_{+1})$  for some function  $\lambda > 0$  and diffeomorphism  $\psi$  of  $S^2$ . By (3.2), the conformal class of  $[\gamma]$  is uniformly bounded and hence the diffeomorphism  $\psi$  is uniformly controlled in  $C^{m+1,\alpha}$  (by  $L$ ) modulo the conformal group. As discussed in Section 2, since  $H$  is bounded away from the constant functions, the control on  $H$  gives control on the conformal group factor. (The conformal group acts properly on the target data  $([\gamma], H)$  when  $H$  is non-constant). It follows that one may precompose  $F_t$  with a bounded curve of diffeomorphisms  $\psi_t$  so that the conformal class is fixed, i.e.  $F_t^* g_{Eucl} = \lambda_t^2 g_{+1}$ . This gives

$$(3.7) \quad \frac{1}{2}\kappa = \delta^* X^T + fA = \varphi\gamma,$$

for some (undetermined) conformal factor  $\varphi = \varphi_t$ . The diffeomorphisms  $\psi_t$  alter  $H_{F_t}$  and  $H'_X$  only by a uniformly bounded factor, which is ignored in the following.

Now compute:

$$\left( \int_{S^2} H_{F_t} dV_{\gamma_t} \right)' = \int_{S^2} H'_X + \int_{S^2} H \frac{1}{2} \text{tr} \kappa = \int_{S^2} H'_X + 2H\varphi.$$

Pairing (3.7) with  $A$  gives

$$\langle A, \delta^* X^T \rangle + f|A|^2 = \varphi H,$$

so that

$$\int_{S^2} \varphi H = \int_{S^2} f|A|^2 + \langle X^T, \delta A \rangle.$$

By the divergence constraint (2.6),  $\delta A = \delta(H\gamma) = -dH$ , and hence

$$\int_{S^2} 2\varphi H = 2 \int_{S^2} f|A|^2 - X^T(H) = -2 \int_{S^2} H'_X,$$

where the last equality follows from (3.6). In sum, for such conformal variations, one has

$$(3.8) \quad \left( \int_{S^2} H \right)' = - \int_{S^2} H'_X,$$

so that

$$(3.9) \quad \left| \left( \int_{S^2} H_{F_t} dV_{\gamma_t} \right)' \right| \leq K \text{area}(F_t),$$

where  $K$  is a bound for  $|H'_X|$ , (cf. the statement following (3.7)). Since  $H$  is uniformly controlled,

$$(3.10) \quad 0 < H_0 \leq H \leq H_0^{-1},$$

integrating over  $t$  gives

$$H_0 \text{area}(F_t) \leq K \int_0^t \text{area}(F_t) + c,$$

which is the same as the differential inequality  $H_0 f' \leq Kf + c$ , for  $f = \int \text{area}(F_t)$ . It follows by a simple calculus argument that

$$\text{area}(F_t) \leq Ce^{K_1 t} \leq Ce^{K_1 L}.$$

■

Next we prove the analog of Proposition 3.1 for the map  $\Pi_1$ .

**Proposition 3.2.** *For  $F \in \mathcal{HImm}_+^{m+1, \alpha}$ , suppose the target data  $([\gamma], [H]) = ([F^*(g_{Eucl})], [H_F])$  for  $\Pi_1$  satisfy*

$$(3.11) \quad ||([\gamma], [H])|| \leq L,$$

*where the norm is taken in the normed target space  $\mathcal{C}^{m, \alpha} \times \mathcal{D}_+^{m-1, \alpha}$ . Then there is a constant  $A_0$ , depending only on  $L$ , such that*

$$(3.12) \quad \text{area}(F) \leq A_0.$$

**Proof:** The proof is essentially the same as that of Proposition 3.1. In this case, the control over the diffeomorphisms  $\psi_t$  comes from the normalization (2.4) giving a slice to the action of the conformal group  $\text{Conf}(S^2)$ . One also needs to show that control over  $[H_{F_t}]$  implies control over  $H_{F_t}$ . To see this, write  $H_{F_t} = H_t + \ell_t$  where  $H_t$  is uniformly controlled (by  $L$ ) in  $\mathcal{C}^{m-1, \alpha}$ . Let  $X_t$  be the conformal vector field generated by the linear function  $x_t$ , where  $\ell_t = a_t + b_t x_t$ . Then by (1.2),

$$(3.13) \quad \int_{S^2} (X_t(H_t) + b_t \varphi) dV_{\gamma_t} = 0,$$

where  $\varphi = X_t(x_t) = \sin r$ ,  $0 \leq r \leq \pi$ , is a fixed function on  $S^2$  up to rotation,  $0 \leq \varphi \leq 1$ , which vanishes only at the poles. Since  $X_t(H_t)$  is uniformly controlled, it follows that the family  $b_t$  is uniformly bounded, so that  $\ell_t$  is also uniformly bounded, which proves the claim. Finally, since  $\min(H + \ell) \geq \min H$  for all normalized affine functions, (3.10) remains valid under the assumption (3.11).

■

The area bounds (3.4) or (3.12) together with (3.1) give an apriori bound on the scale-invariant quantity  $\int |A|^2$ ,

$$(3.14) \quad \int_{S^2} |A|^2 \leq C,$$

for  $H$ -regular branched immersions, with  $C$  depending only on the norm of the target data for  $\Pi_0(F)$  or  $\Pi_1(F)$ .

**Theorem 3.3.** *The maps*

$$(3.15) \quad \Pi_0 : \mathcal{HImm}_+^{m+1,\alpha}(S^2, \mathbb{R}^3) \rightarrow \mathcal{C}^{m,\alpha} \times [C^{m-1,\alpha}(S^2) \setminus \{\text{constants}\}],$$

$$\Pi_0(F) = ([F^*(g_{Eucl})], H_F),$$

and

$$(3.16) \quad \Pi_1 : \mathcal{HImm}_+^{m+1,\alpha}(S^2, \mathbb{R}^3) \rightarrow \mathcal{C}^{m,\alpha} \times \mathcal{D}^{m-1,\alpha},$$

$$\Pi_1(F) = ([F^*(g_{Eucl})], [H_F]),$$

i.e. the maps in (2.26)-(2.27) restricted to  $\mathcal{HImm}_+^{m+1,\alpha}(S^2, \mathbb{R}^3)$ , are smooth proper Fredholm maps.

**Proof:** We first work with  $\Pi_0$ . Suppose  $F_i$  is a sequence in  $\mathcal{HImm}_+^{m+1,\alpha}$  such that the target data  $\Pi_0(F_i) = ([\gamma_i], H_i)$  converge in  $\mathcal{C}^{m,\alpha} \times C^{m-1,\alpha}$  to a limit  $([\gamma], H) \in \mathcal{HImm}_+^{m+1,\alpha}$ . One then needs to show that a subsequence of  $\{F_i\}$  converges in  $C^{m+1,\alpha}$  to a limit map  $F \in \mathcal{HImm}_+^{m+1,\alpha}$ . As in the proof of Proposition 3.1, for simplicity we may assume that the maps  $F_i$  are regular immersions.

It is standard and well-known that if there is a uniform bound for the second fundamental form  $A = A_{F_i}$  of  $\{F_i\}$ , then a subsequence converges in  $C^{1,\alpha}$  to a limit  $C^{1,\alpha}$  immersion  $F$ . (The bound on  $|A|$ , together with control of the diffeomorphisms reparametrizing  $F_i$  as in the proof of Proposition 3.1, imply a bound on the second derivatives of the immersion  $F$ ; the result then follows from the Arzela-Ascoli theorem). Since the data  $([F^*g_{Eucl}], H)$  are elliptic for the map  $F$ , elliptic regularity then shows that the convergence of the target data implies convergence  $F_i \rightarrow F$  in  $C^{m+1,\alpha}$ .

Thus the issue is to understand the structure of  $\{F_i\}$  when  $|A_i|$  blows up as  $i \rightarrow \infty$ . This is done by a blow-up argument. Choose a point  $x = x_i$  on  $S^2$  where  $|A_{F_i}|$  is (locally) maximal, so that if  $|A_{F_i}|(x_i) = \lambda_i$ , then  $\lambda_i \rightarrow \infty$  as  $i \rightarrow \infty$  with

$$|A_{F_i}|(y) \leq \lambda_i,$$

with  $y$  near  $x_i$ . Now rescale the immersion at  $x_i$  by multiplying by  $\lambda_i$ , i.e. consider  $F_i^{\lambda_i} = \lambda_i F_i$ . Here we assume (without loss of generality) that  $F_i(x_i) = 0 \in \mathbb{R}^3$ . We also rescale or blow-up the local coordinates for  $S^2$  near  $x_i$  by  $\lambda_i$ , exactly as in the discussion of the Enneper surfaces in Section 2. Note that norms of derivatives of  $F_i$  are invariant under such simultaneous rescalings of domain and range. One then has

$$(3.17) \quad |A_{F_i^{\lambda_i}}|(y_i) \leq 1, \text{ with } |A_{F_i^{\lambda_i}}|(x_i) = 1.$$

This holds for all  $y_i$  such that  $F_i^{\lambda_i}(y_i)$  is of uniformly bounded distance to  $F_i^{\lambda_i}(x_i) = 0$ . It follows from the constraint equation (2.7) that the Gauss curvature of the blow-up surfaces  $F_i^{\lambda_i}$  remains uniformly bounded.

The standard compactness result used above thus implies that

$$(3.18) \quad F_i^{\lambda_i} \rightarrow F_\infty \text{ in } C_{loc}^{m+1,\alpha},$$

and  $F_\infty$  is an immersion  $F_\infty : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  with  $x_i \rightarrow x_\infty$  and  $F_\infty(x_\infty) = 0$ . Moreover, since

$$H_i^\lambda = \lambda_i^{-1} H_{F_i} \rightarrow 0 \quad \text{as } i \rightarrow \infty,$$

$F_\infty$  is a complete minimal immersion  $\mathbb{R}^2 \rightarrow \mathbb{R}^3$ . Also, since the conformal structure of  $F_i$  is uniformly controlled,  $[F_\infty^*(g_{Eucl})] = [du^2 + dv^2]$  is the standard conformal structure on  $\mathbb{R}^2$ .

The smooth convergence in (3.18) implies that the limit immersion  $F_\infty$  is not totally geodesic, since

$$(3.19) \quad |A_{F_\infty}|(x_\infty) = 1.$$

Moreover, since the  $L^2$  norm of  $A$  is scale invariant, the bound (3.14) holds uniformly for the family  $F_i^{\lambda_i}$  and hence the limit minimal immersion  $F_\infty$  has finite total curvature,

$$\int_{\mathbb{R}^2} |A|_{F_\infty}^2 < \infty.$$

By a well-known result of Osserman, cf. [18] for example, the total scalar curvature  $R = -|A|^2$  of a minimal surface  $\Sigma$  immersed in  $\mathbb{R}^3$  is quantized, i.e.

$$\int_{\Sigma} |A|^2 = 4k\pi,$$

with  $k = 0$  exactly when  $\Sigma$  is a flat totally geodesic plane  $\mathbb{R}^2 \subset \mathbb{R}^3$  while  $k = 1$  exactly for Enneper's surface, (up to scaling).

It follows directly from (3.19) that  $k \geq 1$ , on any sequence  $x_i$  as above in (3.17). In addition, the sequence  $F_i$  satisfies the scale-invariant uniform bound (3.14). Hence there at most

$$N \leq \frac{C}{4\pi},$$

points  $q_j$  (limit points of sequences  $\{x_i\}$ ) where  $|A|$  can blow up.

Away from those points, one has smooth convergence in  $C^{m+1,\alpha}$  to a limit immersion  $F$  - as discussed above in the second paragraph of the proof. The domain of  $F$  here is a finitely punctured two sphere  $S^2 \setminus \cup\{q_j\}$ ,  $1 \leq j \leq N$ . The singular points correspond to formation of branch points at  $F$ . In more detail, near such points  $x_i$ ,  $F_i$  is  $C^{m+1,\alpha}$  close to the the  $\lambda_i^{-1}$ -blow-down  $E_{\lambda_i^{-1}}$  of a complete minimally and conformally immersed plane  $E \simeq \mathbb{R}^2 \rightarrow \mathbb{R}^3$ . Recall as above that the  $C^{m+1,\alpha}$  norm of  $F_i$  is invariant under the rescalings above. The limit  $E_0$  of the minimal immersions  $E_{\lambda_i^{-1}}$  is the map  $h(z) = (z^k, 0) : \mathbb{C} \rightarrow \mathbb{C} \times \mathbb{R} = \mathbb{R}^3$ , i.e. a standard branched immersion with branch point of order  $k - 1 \geq 2$ . From this, it is readily verified that  $F \in \mathcal{HImm}^{m+1,\alpha}$  and  $F_i \rightarrow F$  in  $\mathcal{HImm}^{m+1,\alpha}$ . This proves that  $\Pi$  is proper.

To prove that  $\Pi_1$  is proper, it suffices from the above to prove that if a sequence of immersions  $F_i$  satisfies  $[H_{F_i}] \rightarrow [H]$  then  $H_{F_i}$  converges (in a subsequence) in  $C^{m-1,\alpha}$ . The proof of this is the same as the proof of Proposition 3.2, i.e. as in (3.13), with  $t$  replaced by  $i$ . ■

A proper Fredholm map  $\Pi : \mathcal{B}_1 \rightarrow \mathcal{B}_2$  of index 0 between Banach manifolds has a well-defined degree (mod 2), (the Smale degree) given by

$$(3.20) \quad \deg \Pi = \#\Pi^{-1}(y), \quad (\text{mod } 2),$$

for any regular value  $y \in \mathcal{B}_2$ . The regular values of  $\Pi$  are open and dense in  $\mathcal{B}_2$  and the properness of the map  $\Pi$  ensures that the cardinality in (3.20) is finite. In many situations, the  $\mathbb{Z}_2$ -valued degree can be enhanced to a  $\mathbb{Z}$ -valued degree; one needs suitable orientations for the triple  $\Pi : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ , cf. [5] for example. However, we will not explore this further here.

**Proposition 3.4.** *For the maps  $\Pi_0$  and  $\Pi_1$  in (3.15) and (3.16), one has*

$$(3.21) \quad \deg \Pi_0 = 0,$$

but

$$(3.22) \quad \deg \Pi_1 = 1.$$

**Proof:** It is well-known (and easy to see) that any proper Fredholm map of index 0 between Banach manifolds which is not surjective has degree 0. Clearly  $\Pi_0$  cannot be surjective, exactly due to the obstruction (1.2). This gives (3.21).

To determine  $\deg \Pi_1$ , consider the value  $([\gamma_{+1}], [2])$ , so that  $H = 2 + \ell$  for some affine function  $\ell$  with  $|\ell| \leq 2$ . This corresponds to data for the standard embedding  $\mathbb{S}^2(1) \subset \mathbb{R}^3$ . Clearly the obstruction (1.2) is satisfied for the functions  $H = 2 + \ell$  only for  $\ell = 0$ . By a classical theorem of Hopf, any immersed sphere of constant mean curvature 2 is a reparametrization of the standard embedding. Fixing the conformal class to be  $[\gamma_{+1}]$  implies that the parametrization is conformal, and the 3-point condition (2.4) implies that the mapping is the standard embedding. It follows that the standard embedding uniquely realizes this value for  $\Pi_1$ .

We claim that  $\text{Ker} D\Pi_1 = 0$  at the standard embedding  $F_0$ . Any  $k \in \text{Ker} D\Pi_1$  satisfies

$$(3.23) \quad k_0^T = 0, \quad H'_k = \ell,$$

for some  $\ell$ . As in the proof of Proposition 3.1,  $k = 2\delta^*Z$ , for some vector field  $Z$  along  $F_0$ . Write  $Z = Z^T + fN$ , where  $Z^T$  is tangent to  $\mathbb{S}^2(1)$  and  $N$  is the unit outward normal. Then  $\frac{1}{2}k = \delta^*Z = \delta^*Z^T + fA$ . Since  $A = \gamma$ , the first equation in (3.23) gives

$$\delta^*Z^T = \varphi\gamma,$$

with  $\varphi = \frac{1}{2}\text{div}Z$ . Thus  $Z^T$  is a conformal Killing field on  $\mathbb{S}^2(1)$ . The 3-point normalization (2.4) then forces  $Z^T = 0$ . The vector field  $fN$  is a “Jacobi field” along  $\mathbb{S}^2(1)$ , i.e. satisfies

$$\Delta f + |A|^2 f = \Delta f + 2f = \ell.$$

The same argument as following (2.12) shows that necessarily  $\ell = 0$ . Hence  $f$  is a first eigenfunction of the Laplacian on  $\mathbb{S}^2(1)$ , corresponding to the normal components of Killing fields  $T$  (translations) on  $\mathbb{S}^2(1)$ . However, such infinitesimal translations violate the normalization (2.2) that the immersions  $F$  are based immersions. Thus  $k = 0$  and so  $\text{Ker} D\Pi_1 = 0$ . It follows that  $\deg \Pi_1 = 1$ , which proves the result. ■

The fact that  $\deg \Pi_1 = 1$  implies that  $\Pi_1$  is surjective, which is just the statement of Theorem 1.1. Thus Theorem 1.1 is proved.

**Remark 3.5.** The contrast of the two degrees in (3.21) and (3.22) is rather unusual. Theorem 1.1 gives the existence of a branched immersion  $F : S^2 \rightarrow \mathbb{R}^3$  realizing any prescribed  $([\gamma], [H])$ , so that  $H_F = H + \ell$ , where  $H > 0$  is arbitrarily prescribed. On the other hand, since  $\deg \Pi_0 = 0$ , given one such immersion  $F$  with data  $([\gamma], H_F)$ , there must exist generically at least one more distinct immersion  $F'$ , giving at least two branched immersions realizing  $([\gamma], H_F)$ . Here generic means  $([\gamma], H_F)$  is a regular value of  $\Pi_0$ .

As a concrete example, it follows that for  $\varepsilon$  sufficiently small, the data  $([\gamma], H)$  with  $0 < |H - c| < \varepsilon$  near the standard round spherical data are realized by at least two distinct immersions  $F, F'$ .

Such non-congruent pairs of immersions  $F$  with equal values of  $([\gamma], H)$  may be considered as “conformal Bonnet pairs”. Recall that a Bonnet pair is a pair of immersions which are isometric and with identical mean curvatures. It is well-known that there are no Bonnet pairs of immersions  $S^2 \rightarrow \mathbb{R}^3$ , cf. [14], [20].



#### 4. GENERALIZATIONS

In this section, we discuss several generalizations of Theorem 1.1.

First, many of the results above apply to surfaces  $\Sigma$  of genus  $g > 0$ . In this case, there is no obstruction to the form of the mean curvature  $H$  as in (1.2). This is immediate when  $g > 1$  since such surfaces have no conformal vector fields. In the case of  $g = 1$ , there are conformal vector fields on a torus  $T^2$ , but since they are periodic (or almost periodic) the equation (1.2) does not apriori constrain the form of  $H$  (since the volume form is not determined).

Thus, in the case of higher genus, we ignore the relation (1.2) and the related equivalence relation (1.3). Moreover, the conformal group is always compact in this situation, so there is no need to divide out by this action. Hence, one works directly with the map

$$(4.1) \quad \Pi : \mathcal{H}Imm_+^{m+1,\alpha}(\Sigma, \mathbb{R}^3) \rightarrow \mathcal{C}^{m,\alpha} \times \mathcal{C}_+^{m-1,\alpha},$$

$$\Pi(F) = ([\gamma], H).$$

(There is no need to consider the different cases of  $\Pi_0$  in (3.15) and  $\Pi_1$  in (3.16)).

Let  $\mathcal{M}_c$  be the Riemann moduli space of constant curvature metrics on the surface  $\Sigma$ . Thus  $\gamma_c \in \mathcal{M}_c$  is of constant curvature 0 in case  $\Sigma = T^2$  and of constant curvature  $-1$  in case  $g > 1$ . By the uniformization theorem, any metric  $\gamma$  on  $\Sigma$  is of the form  $\gamma = \lambda^2 \psi^*(\gamma_c)$ , for some diffeomorphism  $\psi$ . An immersion  $F : \Sigma \rightarrow \mathbb{R}^3$  is conformal if  $F^*(g_{Euc})$  is (pointwise) conformal to  $\gamma_c$ , for some  $\gamma_c \in \mathcal{M}_c$ .

It is then straightforward to verify that all of the results of Section 2 hold for  $g > 0$ , so that  $\Pi$  in (4.1) is a smooth Fredholm map, of index 0. The analog of Proposition 3.1 also holds, although the proof requires some further work. The curve of metrics  $\gamma_t$  now has the form  $\gamma_t = \lambda_t^2 \psi_t^*(\gamma_{c(t)})$ , where  $\gamma_{c(t)}$  is a curve in  $\mathcal{M}_c$ . In this situation, the relation (3.7) for the variation of the induced metric must be replaced by

$$\frac{1}{2}\kappa = \delta^* X^T + fA = \varphi\gamma + \bar{\tau},$$

where  $\bar{\tau} = \lambda^2 \tau$  and  $\tau$  is tangent to  $\mathcal{M}_c$ , i.e.  $\tau$  is transverse-traceless with respect to the constant curvature metric  $\gamma_{c(t)} \in \mathcal{M}_c$ . Following the same argument as before, it follows that (3.8) is modified to

$$\left( \int_{\Sigma} H \right)' = - \int_{\Sigma} H'_X - \int_{\Sigma} \langle A, \bar{\tau} \rangle dV_{\gamma_t},$$

so that

$$(4.2) \quad \left| \left( \int_{\Sigma} H \right)' \right| \leq K \text{area}(\Sigma) + \left| \int_{\Sigma} \langle A, \bar{\tau} \rangle \right|.$$

One has

$$\left| \int_{\Sigma} \langle A, \bar{\tau} \rangle \right| \leq \int_{\Sigma} |A|^2 + \int_{\Sigma} |\bar{\tau}|^2.$$

Using (2.7) and the Gauss-Bonnet theorem, the first term on the right is bounded by  $-4\pi\chi(\Sigma) + \int_{\Sigma} H^2 \leq -4\pi\chi(\Sigma) + H_0^{-1} \text{area}(\Sigma)$ , which is of the same form as the first term in (4.2). Next, since  $\gamma_t = \lambda^2 \gamma_{c(t)}$  up to diffeomorphism, one has

$$\int_{\Sigma} |\bar{\tau}|^2 dV_{\gamma_t} = \int_{\Sigma} |\tau|^2 dV_{\gamma_{c(t)}},$$

where the norm and volume form on the left are with respect to  $\gamma_t$  and with respect to  $\gamma_{c(t)} \in \mathcal{M}_c$  on the right. However, this term is bounded, since the curve  $F_t$  has bounded speed and length in the target space  $\mathcal{C}^{m,\alpha}$  and hence in  $\mathcal{M}_c$ . Thus, both the area and the pointwise norm  $|\tau|^2$  are bounded with respect to  $\gamma_{c(t)}$ . It follows that (3.9) again remains valid in this situation, and the proof is completed as before.

The proof of Theorem 3.3 carries over to the higher genus case without change. Thus the map  $\Pi$  in (4.1) is smooth, proper and Fredholm, of index 0.

However, the computation of the degree  $\deg \Pi$  does not carry over, and it is an open question to compute the degree when  $g > 0$ . (The degree may also depend on the component of the space  $\mathcal{HImm}^{m+1,\alpha}(\Sigma, \mathbb{R}^3)$  if this space is not connected).

It would be most natural to compute the degree based on CMC immersed surfaces of higher genus in  $\mathbb{R}^3$ , as done in the case of  $S^2$ . This would require understanding the conformal rigidity and infinitesimal conformal rigidity of such CMC immersed surfaces in  $\mathbb{R}^3$ .

Next, consider conformal immersions with prescribed mean curvature into the simply connected spaces of constant curvature, i.e.  $\mathbb{S}^3$  and  $\mathbb{H}^3$ , up to scaling. Theorem 1.1 generalizes to this setting with only minor changes. For hyperbolic target  $\mathbb{H}^3$ , note that  $H = 2$  is the large-radius limit of the mean curvature of geodesic spheres in  $\mathbb{H}^3$ , (in place of  $H = 0$  in  $\mathbb{R}^3$ ). Thus let  $C_2^{m-1,\alpha}$  be the space of  $C^{m-1,\alpha}$  functions  $H$  on  $S^2$  with  $H > 2$  everywhere and let  $\mathcal{D}_2^{m-1,\alpha}$  be the quotient under the equivalence relation (1.3) as before.

**Theorem 4.1.** *The map*

$$(4.3) \quad \Pi_1 : \mathcal{HImm}_+^{m+1,\alpha}(S^2, \mathbb{S}^3) \rightarrow \mathcal{C}^{m,\alpha} \times \mathcal{D}^{m-1,\alpha},$$

$$\Pi_1(F) = ([F^*(g_{+1})], [H_F]),$$

*is a smooth and proper Fredholm map of index 0 and*

$$\deg \Pi_1 = 1.$$

*Hence Theorem 1.1 holds with target  $\mathbb{S}^3$ .*

*Similarly, for hyperbolic space  $\mathbb{H}^3$ , the map*

$$(4.4) \quad \Pi_1 : \mathcal{HImm}_+^{m+1,\alpha}(S^2, \mathbb{H}^3) \rightarrow \mathcal{C}^{m,\alpha} \times \mathcal{D}_2^{m-1,\alpha},$$

$$\Pi_1(F) = ([F^*(g_{-1})], [H_F]),$$

*is a smooth and proper Fredholm map of index 0 and*

$$\deg \Pi_1 = 1.$$

*Hence Theorem 1.1 holds with target  $\mathbb{H}^3$ .*

**Proof:** It is straightforward to verify that all of the discussion and results in Section 2 carry over to these target spaces. A simple exercise shows that (2.12) remains valid, although it also follows from the invariance of the Fredholm index under continuous deformations.

The basic method of proof of Propositions 3.1 and 3.2 also carries over, with one difference however. Namely, the formula (3.5) for  $H'_X$  is altered by the presence of curvature to

$$H'_X = -\Delta f - (|A|^2 + \text{Ric}(N, N))f + X^T(H).$$

Here  $\text{Ric}(N, N) = 2\kappa$ , where  $\kappa = \pm 1$  according to whether the target is  $\mathbb{S}^3$  or  $\mathbb{H}^3$ . The divergence constraint (2.6) is unaltered, since  $\text{Ric}(N, X^T) = 0$ . As before, one then obtains

$$(4.5) \quad \left( \int_{S^2} H \right)' \leq K \text{area}(F_t) - 4\kappa \int_{S^2} f.$$

Consider first  $\kappa = 1$ , so we are working with the case  $\mathbb{S}^3$ . One has

$$\int_{S^2} f = \int_{S^2} \langle X, N \rangle,$$

where  $N$  is the outward unit normal. Suppose for the moment the immersions  $F_t$  extend to immersions of a 3-ball  $F_t : B^3 \rightarrow \mathbb{R}^3$ . Let  $g_t = F_t^*(g_{+1})$  be the resulting curve of constant curvature  $+1$  metrics on  $B^3$ . Then

$$(4.6) \quad \int_{S^2} \langle X, N \rangle = \frac{d}{dt} \text{vol}(B^3, g_t).$$

Substituting this in (4.5) and integrating as before gives

$$(4.7) \quad \int_{S^2} H \leq K \int_0^t \text{area}(F_t) - 4\text{vol}(B^3, g_t) + c \leq K \int_0^t \text{area}(F_t) + c.$$

Thus, Proposition 3.1 follows as previously.

In general, we may suppose that the initial map  $F_0$  is an embedding and, by a small perturbation, that the normal variation  $f$  of  $F_t$  vanishes only on sets of area zero on  $S^2$ . The embedding  $F_0$  extends to an embedding of the 3-ball  $F_0 : B_1 \rightarrow \mathbb{R}^3$ , with  $F_0^*(g_{+1})$  a constant curvature  $+1$  metric on  $B_1$  inducing the metric  $\gamma$  on  $\partial B_1$ . The smooth family of mappings  $F_t : S^2 \rightarrow \mathbb{R}^3$ ,  $0 \leq t \leq T$ , then gives a map  $F_T : B_{T+1} \rightarrow \mathbb{R}^3$  whose restriction to  $S_{T+1}^2$  is the immersion  $F_t$  - so one has a one parameter smooth family of immersions of the spheres (a regular homotopy). The pullback  $F_T^*(g_{+1})$  is a constant curvature  $+1$ , possibly singular metric on  $B_{T+1}$ , but is regular almost everywhere. The singular set corresponds to the locus where  $f = 0$  and so the volume form vanishes. Hence (4.6) remains valid, and its integrated version in (4.7) holds for all  $t$ . Thus again Proposition 3.1 follows as before.

In the hyperbolic case  $\mathbb{H}^3$ , the same argument as above gives

$$\int_{S^2} H \leq K \int_0^t \text{area}(F_t) + 4\text{vol}(B^3, g_t) + c,$$

where the metrics  $g_t$  are now hyperbolic, i.e. of constant curvature  $-1$ . By a well-known isoperimetric inequality for hyperbolic metrics, cf. [25] for instance,

$$\text{vol}(B^3, g_t) \leq \frac{1}{2} \text{area}(\partial B^3, g_t) = \frac{1}{2} \text{area}(F_t),$$

and hence

$$\int_{S^2} H \leq K \int_0^t \text{area}(F_t) + 2\text{area}(F_t) + c.$$

Since  $H \geq 2 + H_0$  with  $H_0 > 0$ , one can absorb the term on the right into the left and proceed as before.

Thus Proposition 3.1 carries over to both target spaces  $\mathbb{S}^3$  and  $\mathbb{H}^3$ . The proof of the other results and Proposition 3.4 carries over to this setting with only very minor changes which, as before, completes the proof. ■

Finally, the discussion of surfaces of higher genus immersed in  $\mathbb{R}^3$  carries over without further changes to  $\mathbb{S}^3$  and  $\mathbb{H}^3$ . Again, the main remaining question is to compute the associated degree.

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